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RECENT PROGRESS OF THE STUDY OF THE EULER-POISSON EQUATION FOR THE EVOLUTION OF GASEOUS STARS

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§1. Introduction. We are studying the following Euler-Poisson equation:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\rho v_j) = 0 ,$$

(1) $$\rho\left(\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_i} = 0, \quad i=1,2,3,$$

$$\frac{\partial}{\partial t}(\rho S) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\rho S v_j) = 0,$$

(2) $$p = \rho^{\gamma} e^{S},$$

(3) $$\sum_{j=1}^{3} \frac{\partial^2 \Phi}{\partial x_j^2} = 4\pi\rho.$$

Here γ is a constant such that $1 < \gamma \leq 2$. The unknown functions are $\rho = \rho(t, x)$, $v = {}^t(v_1, v_2, v_3) = v(t, x)$, $S = S(t, x)$ and $\Phi =$

$\Phi(t, x)$ of $t \geq 0$ and $x = (x_1, x_2, x_3) \in R^3$. The equations (1)(2)(3) describe the hydrodynamic evolution of the internal structure of a self-gravitating gaseous star. The variable ρ means the density, p the pressure, v the velocity, S the entropy per unit mass and Φ the gravitational potential. The equation (1) is the Euler equation of a compressible fluid without viscosity, the equation (2) is the equation of state for an ideal gas, and the equation (3) is the Poisson equation which determines the external force $-\rho \operatorname{grad}\Phi$ of (1) by the density distribution itself. Since we consider density distributions of compact support, we replace (3) by the Newtonian potential

$$(3) \qquad \Phi(t, x) = - \int_{R^3} \frac{\rho(t, y)}{|x - y|} dy.$$

For a detailed discussion of the equation in the astrophysical context we refer to P. Ledoux and T. Walraven, 6.

The problems relative to the equation (1)(2)(3) are: i) the existence and the uniqueness of the local solution to the Cauchy problem; ii) the existence or non-existence of global solutions; iii) the stability of stationary solutions; etc. The article T. M.,8,1986, was the first to discuss these problems. After this article we continued the study of the Euler-Poisson equation and obtained some results. However our achievement cannot be said to be enough. We expect that more scholars inquire further into this study.

§2. Construction of local solutions. First we discuss the Cauchy problem for (1)(2)(3) under the initial condition

$$(4) \qquad \rho|_{t=0} = \rho^0(x) \geqq 0, \quad v|_{t=0} = v^0(x), \quad S|_{t=0} = S^0(x).$$

We obtained

Theorem 1 (T.M. and S. Ukai, 10, 1987) Let $\rho^0(x)$, $v^0(x)$ *and* $S^0(x)$ *belong to* $C^1(R^3)$, *and* $\rho^0(x) \geqq 0$ *is of compact support. Put*

$$(5) \qquad U^0 = \left((\rho^0)^{\frac{\gamma-1}{2}} e^{\frac{\gamma-1}{2\gamma}S^0},\ v^0,\ S^0 \right).$$

If I) $1 < \gamma \leqq 5/3$ *and* $U^0 \in H^3(R^3)$ *or if II)* $1 < \gamma \leqq 2$, $U^0 \in H^4(R^3)$ *and* $\rho^0 \in H^3(R^3)$, *then there exists a solution* $(\rho, v, S) \in C^1([0, T) \times R^3)$ *of (1)(2)(3)'(4), where* T *is a small positive number.*

The crucial point of the proof of this existence theorem is the integration of the Euler equation (1) for compactly supported density. The standard mathematical treatment of the compressible Euler equation is to transform it to a symmetric hyperbolic system to which Friedrichs-Lax-Kato thory is applicable (3, 4, 5, 7). But in the former study, as in 4, 7, the density ρ was supposed to majorize a positive constant throughout the whole space uniformly. However, in our problem of stars, the density is expected to have a compact support or, at least, to vanish at infinity, for otherwise the Newtonian potential would diverge (Olbers' paradox).

This situation requires other symmetrization than that of 4,7.

Thus we introduced a new variable

$$(6) \qquad w = p^{\frac{\gamma-1}{2\gamma}} = \rho^{\frac{\gamma-1}{2}} e^{\frac{\gamma-1}{2\gamma}S}.$$

Changing the variables from ρ to w, and dividing the second and third equations of (1) by ρ formally, we get a system of the form

$$(7) \qquad A_0(U)\frac{\partial U}{\partial t} + \sum_{j=1}^{3} A_j(U)\frac{\partial U}{\partial x_j} = G(t),$$

where $U = {}^t(w,\ v,\ S)$, $G = {}^t(0,\ -\mathrm{grad}\ \Phi,\ 0)$, $A_0(U)$, $A_j(U)$, $j=1,2,3$, are symmetric matrices and $A_0(U)$ is positive definite uniformly for bounded S. These coefficients are smooth functions of U and there are no problem even when w or ρ vanishes. Then Theorem II of T. Kato, 3, is directly applicable. By this device Theorem 1 can be proved by the standard technique using the Banach's fixed point theorem.

§3. Necessity of improvement of Theorem 1. We cannot content ourselves with the sufficient conditions I) and II) required in Theorem 1. The most serious reason for improvement is as follows. If $6/5 < \gamma \leq 2$, thre exist stationary solutions of the form

$$(8) \qquad \rho = \left(\frac{KA^2\gamma}{4\pi(\gamma-1)}\right)^{\frac{1}{2-\gamma}} \theta(A|x|)^{\frac{1}{\gamma-1}}, \quad v = 0,\ S = \log K,$$

where $\theta(r)$ is the Lane-Emden function, i.e., the solution of

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \theta^{\frac{1}{\gamma-1}} = 0, \quad \theta|_{r=0} = 1, \quad \frac{d\theta}{dr}\Big|_{r=0} = 0$$

(see 1, Chap.IV), and A and K are arbitrary positive constants. These solutions are of class C^1 but not satisfy the conditions of Theorem 1. In other words, the solutions constructed in Theorem 1 are limited to "tame" solutions in the following sense.

Definition. *(ρ, v, S) is called tame solution of (1)(2)(3) on $[0, T)$ if i) $(\rho, v, S) \in C^1([0,T) \times R^3)$, $\rho \geqq 0$, the support of $\rho(t, .)$ is compact, and ii) $\rho^{(\gamma-1)/2} \in C^1([0, T) \times R^3)$, $v \in C([0,T)$; $B(R^3))$, and the additional equations (equations of free fall)*

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial \Phi}{\partial x_i} = 0, \quad i=1,2,3 \tag{9}$$

hold in the exterior of the support of ρ. On the other hand (ρ, v, S) will be called classical solution if only i) is assumed.

Thus the spherically symmetric stationary solution (8) is classical but not tame. We expect, therefore, that the conditions for the local existence theorem will be weakened so as to allow these stationary solutions.

Open Problem 1. To establish the existence theorem of local solutions for a class of initial data which includes the spherically symmetric stationary solutions (8).

§4. Result by P. Gamblin. Recently P. Gamblin proved the following

Theorem 2. (P. Gamblin, 2, 1992) If $1 < \gamma < 9/5$, $U^0 \in H^s_{ul}$ *with* $7/2 < s < (\gamma+1)/(\gamma-1)$ *if* $2/(\gamma-1) \notin N$ *and* $7/2 < s$ *if not, and if* $0 < \rho^0 \in W^{1,p}$ *with* $1 \leqq p < 3$, *then there exists a* C^1*-solution of (1) (2)(3)'(4) locally in time.*

This is a nice result, because the stationary solutions

$$\rho = \left(\frac{3KA^2}{2\pi}\right)^{5/4}(1 + A^2|x|^2)^{-5/2}$$

for $\gamma = 6/5$ satisfy the conditions of this theorem. Therefore this is an answer to Open Problem 1 at least for $\gamma = 6/5$. For the proof, P. Gamblin skillfully uses the results by J. Y. Chemin, S. Alinhac, P. Gérard and J. Rauch concerning the paradifferential calculus of J. M. Bony.

§5. Non-existence of global tame solutions. Triggered by the work 13 of T. Sideris, we studied the question whether tame solutions constructed in Theorem 1 can be continued to $t = +\infty$. We began by considering the equation in which the gravitation is neglected. Let us denote by $(1)_0$ the equation (1) from which the term $-\rho$ grad Φ is dropped. Then we obtained

Theorem 3. (T.M., S. Ukai and S. Kawashima, 9, 1986) Let $(\rho(t), v(t), S(t))$ *be a tame solution of* $(1)_0(2)$ *on* $0 \leqq t < T$. *If the support of* $(\rho(0), v(0))$ *is compact and if* $\rho(0) \not\equiv 0$, *then* T *is finite.*

This theorem claims that any non-trivial tame solution of $(1)_0(2)$ will become not tame after a finite time. But we could not know what will happen actually after that limit time. Thus we have

Open Problem 2. What will happen actualy for a solution at the limit time of the maximal interval of existence as a tame solution?

It was the next task to prove the same conclusion as Theorem 3 for tame solutions of the original equation (1)(2)(3) including the self-gravitation. We began by dealing with spherically symmetric solutions, that is, solutions of the form $\rho = \rho(t, |x|)$, $v = \frac{x}{|x|} V(t, |x|)$, $S = S(t, |x|)$. We obtained

Theorem 4. (T. M. and B. Perthame, 11, 1990) Let $(\rho(t), v(t), S(t))$ *be a spherically symmetric tame solution of (1)(2)(3) on* $0 \leqq t < T$. *If the support of* $(\rho(0), v(0))$ *is compact and if* $\rho(0) \not\equiv 0$, *then* T *is finite.*

We conjecture that this is true for non symmetric tame solutions. Thus we have

Open Problem 3. To prove that the life span of any non trivial tame solution of (1)(2)(3) is finite.

§6. An information about classical solutions. Classical solutions can be global. Then we can ask the asymptotic behavior of global classical solutions. But we know little. We note that along any classical solution of (1)(2)(3)' the total mass $M = \int \rho\, dx$ and the total energy

$$E = \int \left(\frac{1}{2}\rho v^2 + \frac{1}{\gamma-1}p\right)dx - \frac{1}{2}\iint \frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy$$

are independent of t. Computing the second derivative of the function

$$H(t) = \int \rho(t, x)\, x^2\, dx,$$

we get

Theorem 5. (T. M. and B. Perthame, 11, 1990) Suppose $\gamma \geq 4/3$. *Let* $(\rho(t), v(t), S(t))$ *be a global classical solution of (1)(2)(3)'. If* $E > 0$, *then* $\liminf_{t \to +\infty} R(t)/t \geq \sqrt{E/M}$, *where* $R(t) = \sup\{\ |x| :$ $\rho(t, x) \neq 0\ \}$.

§7. Blowing up solutions. On the other hand, we can construct classical solutions which blow up after finite times.

Theorem 6. (T. M.,12, 1992) If $\gamma = 4/3$, then there exists a family of classical solutions of (1)(2)(3) which tend to delta function after finite times in the distribution sense.

In fact we can find particular solutions of the form

$$\rho = \frac{(K/\pi)^{3/2}}{a(t)} y\left(\frac{|x|}{a(t)}\right)^3, \quad v = \frac{\dot{a}(t)}{a(t)} x, \quad S = \log K,$$

where $d^2a/dt^2 = -\lambda/a^2$ and

$$\frac{d^2y}{dr^2} + \frac{2}{r}\frac{dy}{dr} + y^3 = (3/4)(\pi/K^3)^{1/2}\lambda.$$

Choosing $0 \leqq \lambda$ small and $\dot{a}(0) < \sqrt{2\lambda/a(0)}$, we get blowing up solutions since there exists a finite T such that $a(t) \longrightarrow 0$ as $t \longrightarrow T-0$. We can say, therefore, that we have a model of the gravitational collapse of a gaseous star even in the Newtonian (non relativistic) theory. However, this construction depends upon the assumption that γ coincides with the critical exponent $4/3$. Thus we have

Open Problem 4. To construct blowing up classical solutions for γ other than $4/3$.

§8. Conclusion. We have summed up the results concerned with the Euler-Poisson equation we have at the moment. Although we have obtained some results, our achievement should be said to be too slow. In fact the following problem may be open still now.

Open Problem 5. To classify in a strict mathematical way the stability of the spherically symmetric stationary solutions.

Anyway, we believe that the Euler-Poisson equation is enough of a challenge. We expsct that more scholars investigate this equation and that the open problems listed above will be solved in the near future.

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